Mathematics for Computer Science

TD1

September 10^{th}, 2025

Question. *Write the complete proof for*

$$\forall n \in \mathbb{N}, \qquad \Delta_n^2 = \sum_{k=1}^{\Delta_n} (2k-1).$$

Let us first state an intermediate result:

Lemma 1. *For all $m \in \mathbb{N}$,*

$$\sum_{k=1}^{m} (2k-1) = m^2.$$

Proof. (of **Lemma 1**) Let $m \in \mathbb{N}$ and let S_m be the sum wanted, defined as $S_m \stackrel{\text{def}}{=} 1 + 3 + \cdots + (2m-1)$. Since both sums have a finite number of terms, we can write the sum twice and use the commutativity of addition to perform the sum in any convenient order. For all $k \in [\![n]\!]$, combining the k^{th} term of the first sum with the $(m-k+1)^{\text{th}}$ term of the second sum reveals an invariant (*i.e.* a quantity independent of the value of k): each of these combinations adds to $(2k-1) + (2(n-k+1)-1) = 2m$.

$$\begin{array}{ccccccccccccc}
S_m & = & 1 & + & 3 & + & \cdots & + & (2k\text{-}1) & + & \cdots & + & (2m\text{-}1) \\
S_m & = & (2m\text{-}1) & + & (2m\text{-}3) & + & \cdots & + & 2(m\text{-}k\text{+}1)\text{-}1 & + & \cdots & + & 1 \\
\hline
2S_m & = & \multicolumn{11}{c}{\underbrace{2m \quad + \quad 2m \quad + \quad \cdots \quad + \quad 2m \quad + \quad \cdots \quad + \quad 2m}_{m \text{ terms}}}
\end{array}$$

Since adding the sum S_m twice amounts to adding $2m$, m times, we have $2S_m = 2m \times m = 2m^2$. In particular, we obtain the value of the sum:

$$S_m = \sum_{k=1}^{m} (2k-1) = m^2.$$

$\square$

We are now ready to prove the wanted result.

Proof. From this more general result, we can derive our desired identity as a corollary of **Lemma 1**: let $n \in \mathbb{N}$. Applying **Lemma 1** for $m \stackrel{\text{def}}{=} \Delta_n \in \mathbb{N}$, we have indeed:

$$\Delta_n^2 = \sum_{k=1}^{\Delta_n} (2k-1).$$

$\square$